Persistent Homology

Feng Ling

The University of Texas At Austin

FLing@utexas.edu

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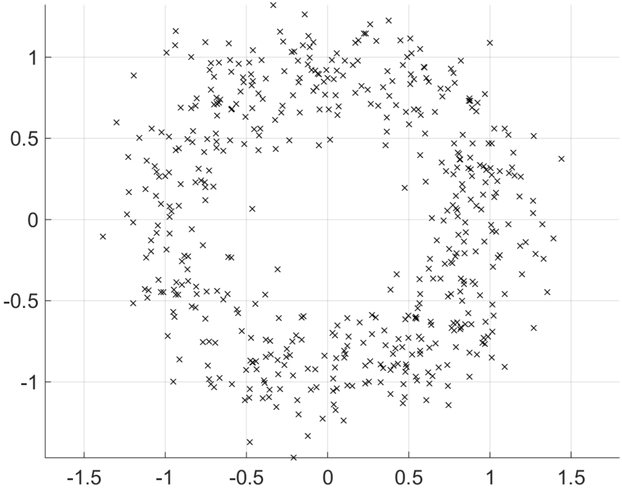
Shape of Data

What is this?



Shape of Data

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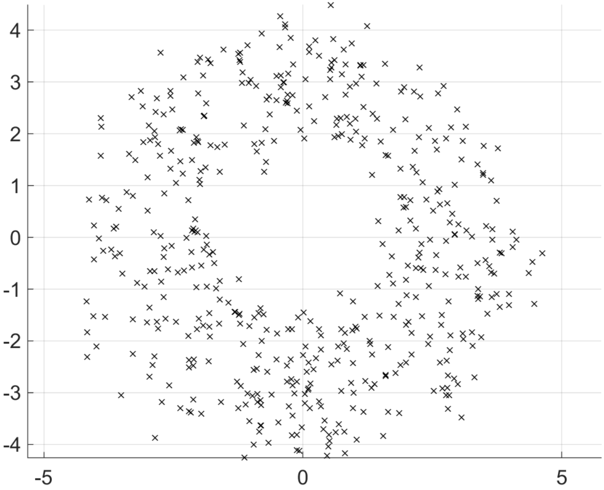

Shape of Data

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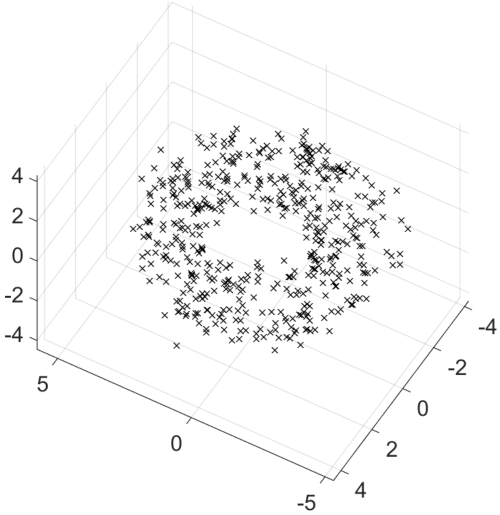
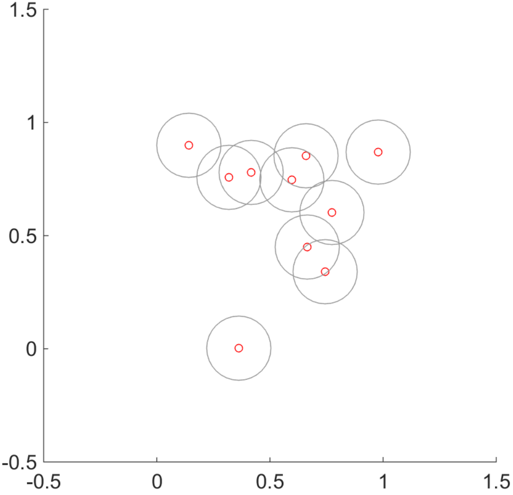


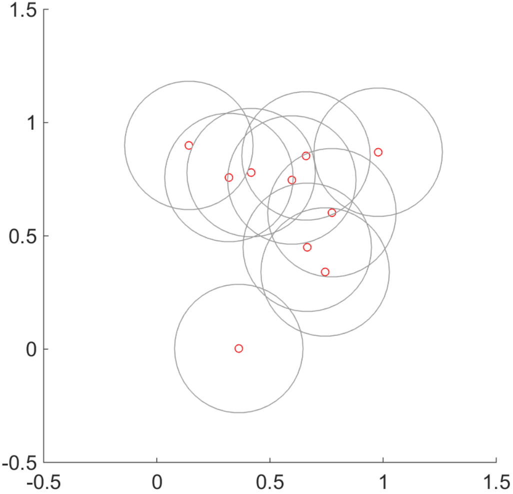
Čech, Vietoris-Rips

A naive strategy is to look at neighborhoods (closed balls) of the point cloud for the underlying geometry and topology...



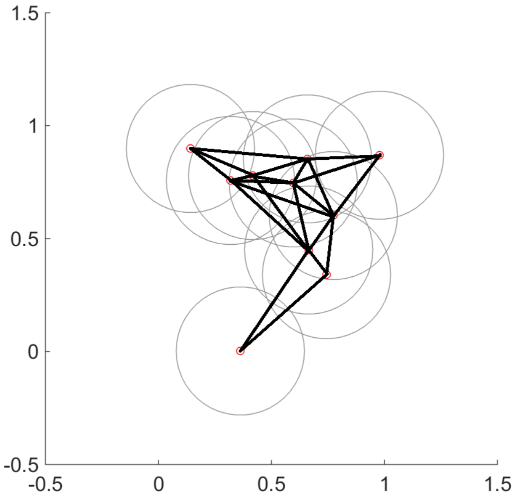
Čech, Vietoris-Rips

By changing the size of our balls, we can obtain different relation between points through intersections of these neighborhood balls...



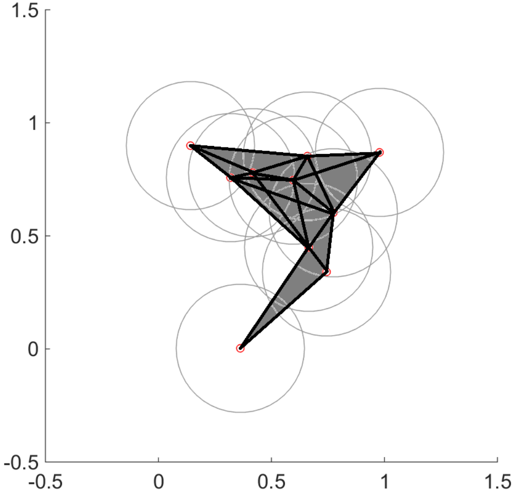
Čech, Vietoris-Rips

Connecting points that have intersecting neighborhood balls, we get a simplicial complex for each radius r. This gives us the "Čech complex."



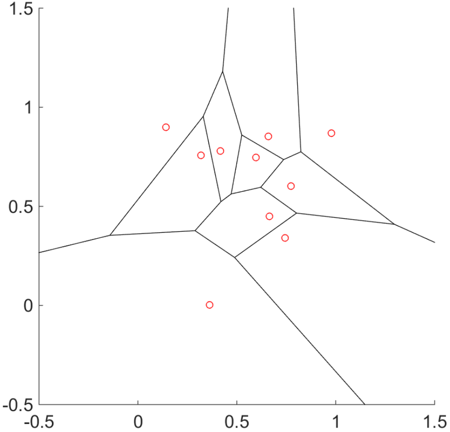
Čech, Vietoris-Rips

Filling in the "spaces" between the edges whenever possible, we obtains the so-called "Vietoris-Rips complex."



Voronoi, Delaunay, Alpha

As you can see there are non-planar complications even in our 10 point 2D case. So let's try to simplify it without losing too much information. One easy way is through intersecting with Voronoi diagrams and its nerves, the Delaunay complexes.



Voronoi, Delaunay, Alpha

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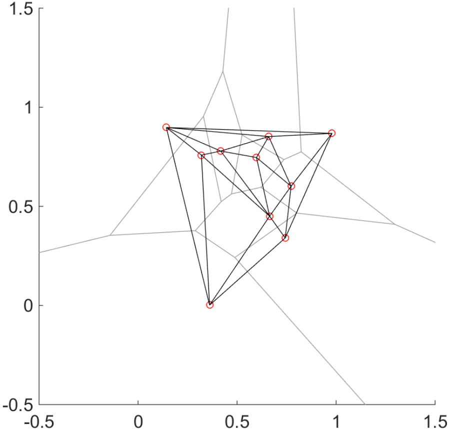
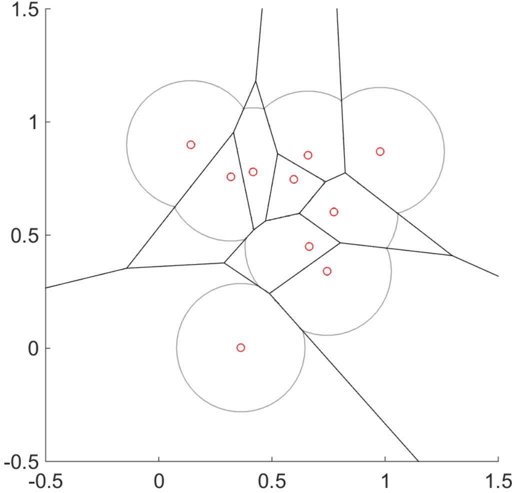


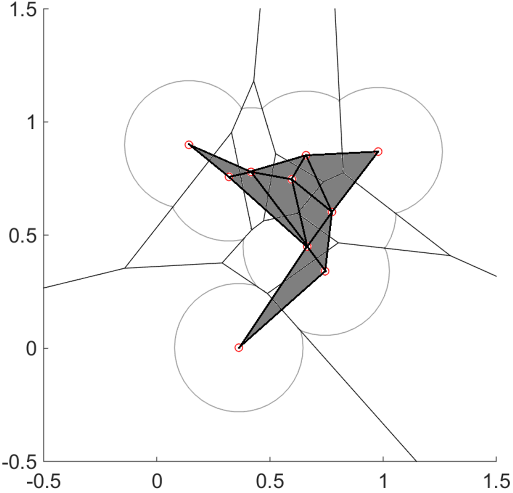
Voronoi, Delaunay, Alpha

Then we get the following, given a fixed radius of neighborhood balls...



Voronoi, Delaunay, Alpha

Filling in we get the so-called “Alpha Complex.”



Weighted Alpha, Witness, and Beyond

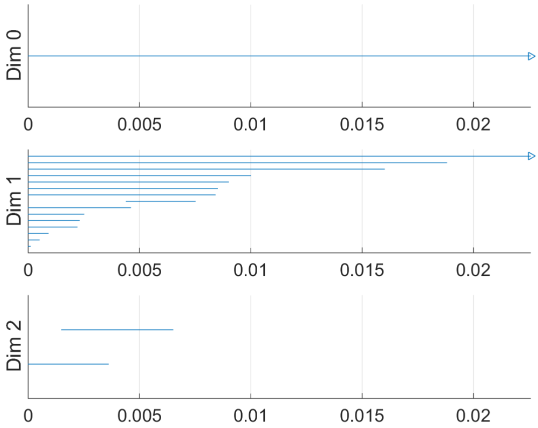
We can do more for realistic representation of point cloud data or improving computational efficiencies.
One is weighted Alpha complex, where although the size of all neighborhood balls change with a single parameter, they are scaled differently at each point.
Another is instead of checking all edges to fill in higher dimensional simplices, we can use witness points to expand, and other points to fill.
There are also more complicated stuff.

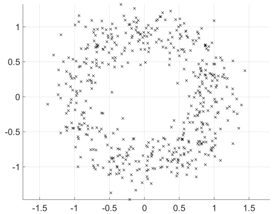
Persistence of Homology Classes

One way to define (singular) homology classes is that they are the cycles that are not boundaries of higher dimensional cells up to homotopy (continuous deformation). It measures connectivity/holes in different dimensions.
Notice that since our complex depends on one parameter - the radius of our chosen neighborhood (so we have a filtration). And **how long a homology class "persists" as our radius parameter increase** from 0 to infinity is what we measure.
Limiting cases: we started out with 0-dimensional manifolds, and end up with a single connected closed component of $\mathbb{R}^n$.
Birth and Death of these classes can be treated rigorously as points on persistence diagrams.

Practical uses

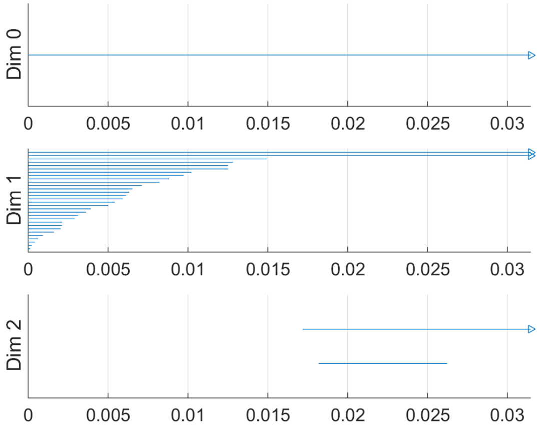
To resolve our teaser problems using JavaPlex...

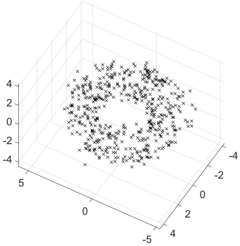




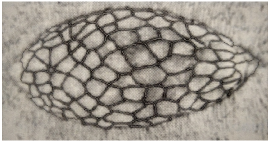
Practical uses

And voila!





Advanced applications

Image segmentation, 

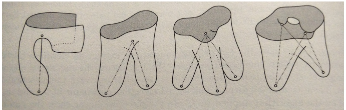
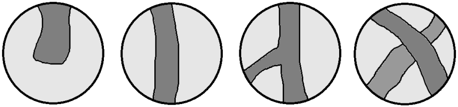
Protein docking/folding analysis, 

Figure IX.7: From left to right: a one-, two-, three-, and four-legged elevation maximum of a surface embedded in 3-dimensional space.



Root structure reconstruction,
Natual image classifications (image compression)...

References

Herbert Edelsbrunner and John Harer (2010)
Computational Topology

Gunnar Carlsson et. al. (2008)
OntheLocalBehaviorofSpacesofNaturalImages

And most importantly my mentor: Ahmad!