(10) ~~Title of this talk is (co) fiber seq.~~ ~~they are important tools in htpy theory, a kind of mathematicians used to study shapes of spaces.~~

Goal of this talk is to use tools called (co)fiber seq. to show the surprising ~~demonstrate how these tools can be used to obtain the classical~~ fact that there is a 3-dim "hole" in the 2 sphere.

[time constrain. no ~~basic~~ proofs]

1st we want to say that we work in a ~~kind~~ class category of spaces called ~~based~~ topological based spaces. there are just your everyday nice topological spaces with a distinguished pt. we write usually as $*$. X like circles.

$X, \exists *_X \in X$.

In this category. the natural maps ~~between spaces~~ are the based maps where we require our special pt ~~always~~ go to ~~the~~ special point.

Def: $f: X \to Y$ is based if $f(*_X) = *_Y$.

we denote the maps from X to Y based spaces as $F(X, Y)$ & luckily, this is also a based space with cst base point map as its base pt

~~There~~ we have an interesting adjuction. (prev. talk)

write this backwards.

$$F(X \wedge -, Y) \cong F(X, F(\overline{\gamma}, Y)).$$

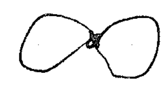
homeom.

~~~~ we know what RHS is. maps from space X to space of maps $-, Y$.

the LHS. is called a smash product.

Def: $X \wedge Y := X \times Y / X \vee Y$ we use this to denote two spaces joined at their basepoint.

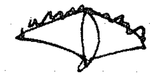
Cartesian product. Contracting quotienting the Right spaces.



given this def. the adjuncton can be checked "easily". (took me a whiles tho.) "trivially".

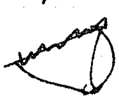
Next a few more definitions:

$\Sigma X = X \wedge S^1$ ← circle.



$\Omega X = F(S^1, X)$ space of loops in X. ~~~~

$CX = X \wedge I / X \times \{0\}$ ← interval



$PX = F(I, X)$. Space of paths in X

You will notice that the interval & circle shows up repeated b/c turns out these are really the 2 kind of spaces that we
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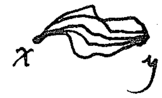
(a)

Finally Def: $[X;Y] := F(X,Y) \,/\,$cts deformation.



(for advanced, same as pass thru π_0 functor) or self

ask. Ernie.

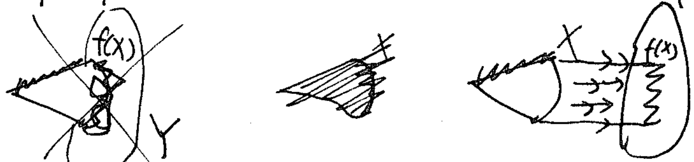
(b) With these definitions settled, we want to talk about the main subject of this talk: cofiber & fiber sequences.

given $f: X \to Y$. in our world/category. we have two sequences.

$$\left(\cdots \to \Omega X \to \Omega Y \to Ff \xrightarrow{\pi} \left(X \xrightarrow{f} Y \right) \xrightarrow{i} Cf \to \Sigma X \to \Sigma Y \to \cdots \right)$$

~~Ask~~ ☆: say it's 2 seq. & left keep applying.
note2self.

Here Cf — cofiber
$CX \cup_f Y$



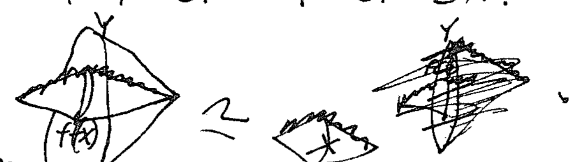
Ff — fiber unfortunately no good way to draw pictures so

$X \times_f PY = \{ (x,\gamma) \in X \times PY \mid f(x) = \gamma(1) \}$.
cartesian prod

the reason we think these sequences are special b/c we can see it as performing the same operation. over & over again.

for cofiber side: claim is ~~X → Y → Cf~~ :: using sat notation

$X : Y : Cf :: Y : Cf : \Sigma X$.



& one can prove this relation propagates indefinitely

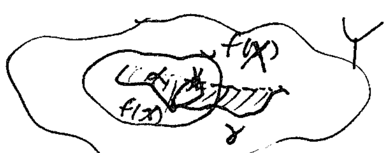
for fiber side: we have

$Ff : X : Y :: \Omega Y : Ff : X$.

ask ernie so is homotopy eq. only.

b/c. $\gamma \in \Omega Y$, $\gamma(0) = \gamma(1) = *_Y$.

$(Ff) \times_\pi PX = \{ ((x,\gamma), \alpha) \mid f(x) = \gamma(1),\ \pi(x,\gamma) = \alpha(1) \}$
$\in PX$
$\Updownarrow$
$x = \alpha(1)$.



again hmtpy eq. (ask ernie)

again one can prove this relation propagates this time to the left indefinitely.

Next the amazing fact is that these (co)fibre seq. give rise to the following long exact sequence. (in case cat person did not see)

For those who don't know what exact sequences means.
it is ⓐ a ~~sequence~~ chain of objects with maps from one to next.
ⓑ. the maps are "memoryless" ~~(maybe markov analogy)~~
as in $\cdots \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to \cdots$

$g \circ f(X') \Leftrightarrow *_{X''}$ things 2 step ago are ~~trivial~~ exactly.

insert ⓐ after.

$$[X, Z] \xrightarrow{f^*} [Y, Z] \to [Cf, Z] \to [\Sigma X, Z] \to \cdots$$

$$\to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y]$$

now we shift gears to explain ~~briefly~~ what is a fiber bundle.

$F \to E \downarrow p\; B$ $\quad E|_U = U \times F$. E are spaces that locally (open set) looks like simple cartesian products

this construction is ~~the example~~ relevant to us ~~b/c~~ example $S^1 \to S^3 \to S^2$ the Hopf Bundle. many way to prove it is intact

~~these~~ turns out conviently for us, $F \to E \xrightarrow{p} B$
is ~~one realizable~~ htpyically same as $Fp \to E \xrightarrow{p} B$.

which means with other lemmas omitted that applying the LES above:

$$\to [Z, \Omega E] \to [Z, \Omega B] \to [Z, F] \to [Z, E] \to [Z, B]$$

$\quad\cong \qquad\qquad \cong \qquad$ use adjunction.

$$[\Sigma Z, E] \to [\Sigma Z, B] \to \cdots\cdots$$

substitute $Z = S^1$ $F = S^1$, $E = S^3$, $B = S^2$ yields

$$\to [S^3, S^1] \to [S^3, S^3] \to [S^3, S^2] \to [S^2, S^1] \to [S^2, S^3] \to \cdots$$

(now we know from $\varphi: \mathbb{R} \to S^1 \subset \mathbb{C}$, $t \mapsto \exp(2\pi i t)$. & $\mathbb{R} \simeq I \simeq *$. contractible.
that $[S^n, S^1] = \cancel{0}\, t$. if $n \neq 1$.)

switch.

& heuristically it is pretty easy to believe $[S^n, S^1] \cong \mathbb{Z}$