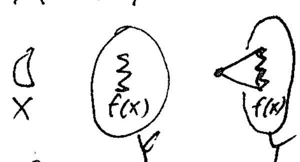
goal of this talk is to ~~use some tools called~~ explain briefly what are cofiber ~~seq.~~ & fiber seq.
& how can one use them to obtain the surprising result that there's a "3-dim" hole in the 2-sphere. so. say we are trying to study spaces. ~~we are~~ the easiest way to get information about unknown spaces is to figure out properties of maps from spaces that we know a lot about to it or from it to ones we know about.

This brings out that the key concept that we need to study are maps between spaces. say given $X \xrightarrow{f} Y$ "nice" top. sp. the easiest kinds of maps are "nice" inclusions & surjections. But an arbitrary map like f need not be either inj. or surj. so we will construct the so-called cofiber sequences to study how similar f is to a nice simple closed inclusion, & then the fiber sequence for how close is f ~~to~~ a surjection.

A cofiber sequence consists of $X \xrightarrow{f} Y \xrightarrow{i} Cf$ where Cf is the cofiber. where i is the inclusion map

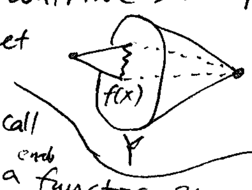
we can draw these pictorially as



in a sense. we are trying to get something like "Y/X" where we quotient out the image of X in Y via contracting it to a point slowly outside.

this sequence is special because we can continue such process: $Y \xrightarrow{i} Cf \to Ci \to \cdots$

here if we perform the same procedure we get



it is interesting to note that this is $\simeq_{\text{htpy}}$

$X \cong f(X)$ if we think of contracting the outside.

the picture on the right is an object that we call the suspension of X, ΣX. turns out this is a endofunctor & we can continue this sequence forever as
$$\to \Sigma X \to \Sigma Y \to \Sigma Cf \to \Sigma(\Sigma X) \to \cdots$$

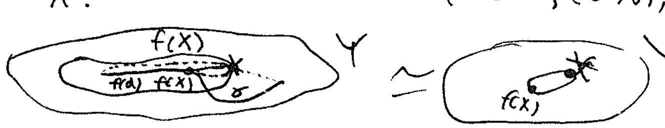
$\boxed{\text{maps}(\Sigma X, Y) = \text{maps}(X, \Omega Y)}$. with picture.

useful fact to keep in mind here is $\Sigma S^n = S^{n+1}$

for the fiber sequence, the surjective side, we have $Ff \xrightarrow{\pi} X \xrightarrow{f} Y$. unfortunately there's not a good way to draw pictures as before, bc we can work in coordinates as by definition $Ff = \{(x,\gamma) \in X \times PY \mid f(x) = \gamma(1)\}$. Here γ is a path in the space Y. & so the relation here means that the path γ ends inside the image of X in Y.

Now we can also extend this sequence in the ~~different~~ direction to the left. to the left. $F\pi \to Ff \xrightarrow{\pi} X$. Now here we have $F\pi = \{((x,\gamma),\alpha) \in Ff \times PX \mid \underbrace{\pi(x,\gamma)}_{x} = \alpha(1)\}$ b/c π is the projection.

pictorially, $\forall x$



a loop in Y.

we write this as ΩY. there the claim is we have $\cdots \to \Omega\Omega Y \to \Omega Ff \to \Omega X \to \Omega Y \to \cdots$
notice that these two sequences looks very much like dual of each other. & in fact we do have

The main reason why they are useful is that we can obtain algebraic statements out of it. Now focusing on the fiber sequence, we have a "long exact sequence" ~~homotopy groups~~
$$\cdots \to [Z,\Omega Ff] \to [Z,\Omega X] \to [Z,\Omega Y] \to [Z,Ff] \to [Z,X] \to [Z,Y]. \quad \forall Z \text{ - a nice space.}$$

here we note that in fact is Z satisfies a certain condition ($\underline{Z = \Sigma\Sigma W}$).

we infact have the above a LES of abelian groups. here we know exactly what exactness mean: $\ker g = \operatorname{im} f$. for $\xi' \xrightarrow{f} \xi \xrightarrow{g} \xi''$

now finally to get our desired result, I will introduce a map $\underline{S^3 \xrightarrow{P} S^2}$ called "Hopf bundle" or "Hopf fibration" and ~~one can prove~~ there are many ways to prove that in fact. $F_P \to S^3 \xrightarrow{P} S^2$. this relation plugged in above yields.
$$F_P \cong S^1$$

$$\to [Z, \Omega S^1] \to [Z, \Omega S^3] \to [Z, \Omega S^2] \to [Z, S^1] \to [Z, S^3] \to [Z, S^2]$$

invoking the adjunction earlier $\operatorname{map}[\Sigma X, Y] \cong \operatorname{map}(X, \Omega Y)$.

& plug in $Z = S^2$.

$$\to [\Sigma S^2 = S^3, S^1] \to [\Sigma S^2 = S^3, S^3] \to [S^3, S^2] \to [S^2, S^1] \to [S^2, S^3] \to [S^2, S^2]$$

Now we need two facts $[S^n, S^n] = \pi_n(S^n) \cong \mathbb{Z}$. via ~~wrapping~~ the degree map.

(i.e. we can wrap the sphere around k times $\forall k \in \mathbb{Z}$.)

$[S^n, S^1] \cong 0 \quad \forall n \neq 1$. via $\varphi: \mathbb{R} \to S^1 \subset \mathbb{C}$.
$t \mapsto \exp(2\pi i t)$.

b/c $\mathbb{R}$ is contractible, as the circle S^1's universal cover covering space theory dictates that. a map to S^1 can be lifted to $\mathbb{R}$ & contractibility imply you can continuously deform the map to the constant map.

then we get
$$\to 0 \to \mathbb{Z} \to [S^3, S^2] \to 0 \to \cdots$$

$\Rightarrow [S^3, S^2] \cong \mathbb{Z}$. in usual notations $\pi_3(S^2) \cong \mathbb{Z}$.

this shows that there is somehow a whole class of nontrivial map from 3-sphere to 2-sphere.

demonstrating how 2-sphere has a 3-dim "hole".